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Capacities in Wiener space, quasi-sure lower functions, and Kolmogorov's ε-entropy

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Abstract

We propose a set-indexed family of capacities $\{\mathrm{cap}_G\}_{G\subseteq\mathbf{R}_+}$ on the classical Wiener space $C(\mathbf{R}_+)$. This family interpolates between the Wiener measure ($\mathrm{cap}_{\{0\}}$) on $C(\mathbf{R}_+)$ and the standard capacity ($\mathrm{cap}_{\mathbf{R}_+}$) on Wiener space. We then apply our capacities to characterize all quasi-sure lower functions in $C(\mathbf{R}_+)$. In order to do this we derive the following capacity estimate which may be of independent interest: There exists a constant $a > 1$ such that for all $r > 0$,

$$\frac{1}{a}\mathrm{K}_G(r^6)\exp\left(-\frac{\pi^2}{8r^2}\right) \le \mathrm{cap}_G\{f^\star \le r\} \le a\mathrm{K}_G(r^6)\exp\left(-\frac{\pi^2}{8r^2}\right).$$

Here, K_G denotes the Kolmogorov ε-entropy of G, and $f^\star := \sup_{[0,1]}|f|$.

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1. Introduction

Let $C(\mathbf{R}_+)$ denote the collection of all continuous functions $f : \mathbf{R}_+ \to \mathbf{R}$. We endow $C(\mathbf{R}_+)$ with its usual topology of uniform convergence on compacts as well as the corresponding Borel σ-algebra $\mathscr{B}$. In keeping with the literature, elements of $\mathscr{B}$ are called *events*.

Denote by μ the Wiener measure on $(C(\mathbf{R}_+), \mathscr{B})$. Recall that an event Λ is said to hold almost surely [a.s.] if $\mu(\Lambda) = 1$.

Next we define $U := \{U_s\}_{s\geq 0}$ to be the *Ornstein–Uhlenbeck process* on $C(\mathbf{R}_+)$. The process U is characterized by the following requirements:

(1) It is a stationary infinite-dimensional diffusion with value in $C(\mathbf{R}_+)$;
(2) Its invariant measure is μ. This implies that for any fixed $s \geq 0$, $\{U_s(t)\}_{t\geq 0}$ is a standard linear Brownian motion.
(3) For any given $t \geq 0$, $\{U_s(t)\}_{s\geq 0}$ is a standard Ornstein–Uhlenbeck process on $\mathbf{R}$; i.e., it satisfies the stochastic differential equation,

$$\mathrm{d}U_s(t) = -U_s(t)\mathrm{d}s + 2^{1/2}\mathrm{d}X_s \quad \text{for all } s \geq 0, \tag{1.1}$$

where X is a Brownian motion.

Following Malliavin [12], we say that an event Λ holds *quasi-surely* [q.s.] if

$$\mathrm{P}\{U_s \in \Lambda \text{ for all } s \geq 0\} = 1. \tag{1.2}$$

Because $t \mapsto U_s(t)$ is a Brownian motion, any event Λ that holds q.s. also holds a.s. The converse is not always true. For example, define Λ_0 to be the collection of all functions $f \in C(\mathbf{R}_+)$ that satisfy $f(1) \neq 0$ [5]. Evidently, Λ_0 holds a.s. because with probability one Brownian motion at time one is not at the origin. On the other hand, Λ_0 does not hold q.s. because $\{U_s(1)\}_{s\geq 0}$ is point recurrent. So the chances are 100% that $U_s(1) = 0$ for some $s \geq 0$.

Despite the preceding disclaimer, a number of interesting classical events of full Wiener measure do hold q.s. A notable example is a theorem of Fukushima [5]. We can state it, somewhat informally, as follows:

$$\text{The Law of the Iterated Logarithm (LIL) of Khintchine [7] holds q.s.} \tag{1.3}$$

It might help to recall Khintchine's theorem: For μ-every $f \in C(\mathbf{R}_+)$,

$$\limsup_{t\to\infty} \frac{f(t)}{(2t \ln\ln t)^{1/2}} = 1. \tag{1.4}$$

Thus we are led to the precise formulation of (1.3): With probability one, the continuous function $f := U_s$ satisfies (1.4), simultaneously for all $s \geq 0$.

For another example consider "the other LIL" which was discovered by Chung [1]. Chung's LIL states that for μ-almost every $f \in C(\mathbf{R}_+)$,

$$\liminf_{t\to\infty} \frac{\sup_{u\in[0,t]} |f(u)|}{(t/\ln\ln t)^{1/2}} = \frac{\pi}{8^{1/2}}. \tag{1.5}$$

Fukushima's method can be adapted to prove that

$$\text{Chung's LIL holds q.s.} \tag{1.6}$$

To be more precise: With probability one, the continuous function $f := U_s$ satisfies (1.5) simultaneously for all $s \geq 0$.

Mountford [15] has derived the quasi-sure integral test corresponding to (1.3). One of the initial aims of this article was to complement Mountford's theorem by finding a precise quasi-sure integral test for (1.6). Before presenting this work, let us introduce the notion of "relative capacity". A finite-dimensional version of these relative capacities appeared implicitly in a beautiful characterization, due to [6], of when semipolar sets are polar for nice Lévy processes.

For all Borel sets $G \subseteq \mathbf{R}_+$ and $\Lambda \in C(\mathbf{R}_+)$ define

$$\mathrm{cap}_G(\Lambda) := \int_0^\infty \mathrm{P}\{U_s \in \Lambda \text{ for some } s \in G \cap [0, \sigma]\}\, \mathrm{e}^{-\sigma}\, \mathrm{d}\sigma. \tag{1.7}$$

We think of $\mathrm{cap}_G(\Lambda)$ as the *capacity of Λ relative to the coordinates in G*. The special case $\mathrm{cap}_{\mathbf{R}_+}$ is well known and well studied [5]; $\mathrm{cap}_{\mathbf{R}_+}$ is called *the capacity on Wiener space*. According to (1.2), an event Λ holds q.s. if and only if its complement has zero $\mathrm{cap}_{\mathbf{R}_+}$ capacity.

The case where $G := \{s\}$ is a singleton is even better studied because of the simple fact that $\mathrm{cap}_{\{s\}}$ is a multiple of the Wiener measure. Thus, $G \mapsto \mathrm{cap}_G(\Lambda)$ interpolates from the Wiener measure ($G = \{0\}$) to the standard capacity on Wiener space ($G = \mathbf{R}_+$). This "interpolation" property was announced in the Abstract.

Now let $H : \mathbf{R}_+ \to \mathbf{R}_+$ be decreasing and measurable, and define

$$\mathscr{L}(H) := \left\{ f \in C(\mathbf{R}_+) : \liminf_{t\to\infty} \left[\sup_{u\in[0,t]} |f(u)| - t^{1/2} H(t) \right] > 0 \right\}. \tag{1.8}$$

A decreasing measurable function $H : \mathbf{R}_+ \to \mathbf{R}_+$ is called an *a.s.-lower function* if $\mathscr{L}(H)$ holds a.s.; i.e., μ-almost every $f \in C(\mathbf{R}_+)$ is in $\mathscr{L}(H)$. Likewise, H is called a *q.s.-lower function* if $\mathscr{L}(H)$ holds q.s. [The literature actually calls the function $t \mapsto t^{1/2}H(t)$ an a.s. [q.s.] lower function if $\mathscr{L}(H)$ holds a.s. [q.s.], but we find our parameterization here convenient.]

To understand the utility of these definitions better, consider the special case that $H(t) = (c/\ln\ln t)^{1/2}$ for a fixed $c > 0$ ($t \geq 0$). In this case, Chung's LIL (1.5) states that $\mathscr{L}(H)$ holds a.s. if $c < \pi/8^{1/2}$; its complement holds a.s. if $c > \pi/8^{1/2}$. In fact, a precise P-a.s. integral test is known [1]; see Corollary 1.3.

We aim to characterize exactly when $(\mathscr{L}(H))^c$ has positive cap_G-capacity. Define K_G to be the *Kolmogorov ε-entropy* of G [3,16]; i.e., for any $\varepsilon > 0$, $k = \mathrm{K}_E(\varepsilon)$ is the maximal number of points $x_1, \ldots, x_k \in E$ such that whenever $i \neq j$, $|x_i - x_j| \geq \varepsilon$.

Theorem 1.1. *Choose and fix a decreasing measurable function $H : \mathbf{R}_+ \to \mathbf{R}_+$, and a bounded Borel set $G \subset \mathbf{R}_+$. Then, $\mathrm{cap}_G((\mathscr{L}(H))^c) = 0$ if and only if there exists a decomposition $G = \cup_{n=1}^\infty G_n$ in terms of closed sets $\{G_n\}_{n=1}^\infty$, such that*

$$\int_1^\infty \frac{\mathrm{K}_{G_n}(H^6(s))}{sH^2(s)} \exp\left(-\frac{\pi^2}{8H^2(s)}\right) \mathrm{d}s < \infty \quad \textit{for all } n \geq 1. \tag{1.9}$$

Theorem 1.1 yields the following definite refinement of (1.5).

Corollary 1.2. *Choose and fix a decreasing measurable function $H : \mathbf{R}_+ \to \mathbf{R}_+$. Then, $\mathscr{L}(H)$ holds q.s. if and only if*

$$\int_1^\infty \exp\left(-\frac{\pi^2}{8H^2(s)}\right) \frac{\mathrm{d}s}{sH^8(s)} < \infty. \tag{1.10}$$

Theorem 1.1 also contains the original almost-sure integral test of Chung [1]. To prove this, simply plug $G = \{u\}$ in Theorem 1.1. Then, $\mathrm{K}_{\{u\}\cap J}(\varepsilon)$ is one if $u \in J$ and zero otherwise. Thus we obtain the following.

Corollary 1.3 (*[1]*). *Choose and fix a decreasing measurable function $H : \mathbf{R}_+ \to \mathbf{R}_+$. Then $\mathscr{L}(H)$ holds a.s. if and only if*

$$\int_1^\infty \exp\left(-\frac{\pi^2}{8H^2(s)}\right)\frac{\mathrm{d}s}{sH^2(s)} < \infty. \tag{1.11}$$

To put the preceding in perspective define

$$H_\nu(t) := \frac{\pi}{|8\,(\ln_+\ln_+ t + \nu \ln_+\ln_+\ln_+ t)|^{1/2}} \qquad \text{for all } t, \nu > 0, \tag{1.12}$$

where $\ln_+(x) := \ln(x \wedge 0)$ and $\ln(0) := -\infty$. $[1/0 := \infty]$ Then, we can deduce from Corollaries 1.2 and 1.3 that $\mathscr{L}(H_\nu)$ occurs q.s. if and only if $\nu > 5$, whereas $\mathscr{L}(H_\nu)$ occurs a.s. if and only if $\nu > 2$. In particular, $\mathscr{L}(H_\nu)$ occurs a.s. but not q.s. if $\nu \in [2, 5)$. The following is another interesting consequence of Theorem 1.1.

Corollary 1.4. *Let $G \subseteq [0, 1]$ be a non-random Borel set. Then,*

$$\begin{aligned} \dim_{\mathrm{P}} G > \frac{\nu-2}{3} &\Longrightarrow \mathrm{cap}_G\left((\mathscr{L}(H_\nu))^c\right) > 0, \quad \textit{whereas} \\ \dim_{\mathrm{P}} G < \frac{\nu-2}{3} &\Longrightarrow \mathrm{cap}_G\left((\mathscr{L}(H_\nu))^c\right) = 0. \end{aligned} \tag{1.13}$$

Here, $\dim_{\mathrm{P}} G$ denotes the packing dimension [13] *of the set G.*

Throughout this paper, uninteresting constants are denoted by a, b, α, A, etc. Their values may change from line to line.

2. Brownian sheet, and capacity in Wiener space

We will be working with a special construction of the process U. This construction is due to Williams ([14], Appendix).

Let $B := \{B(s,t)\}_{s,t\geq 0}$ denote a two-parameter Brownian sheet. This means that B is a centered, continuous, Gaussian process with

$$\mathrm{Cov}\left(B(s,t), B(s',t')\right) = \min\left(s,s'\right) \times \min\left(t,t'\right) \quad \text{for all } s, s', t, t' \geq 0. \tag{2.1}$$

The Ornstein–Uhlenbeck process $U = \{U_s\}_{s\geq 0}$ on $C(\mathbf{R}_+)$ is precisely the infinite-dimensional process that is defined by

$$U_s(t) = \mathrm{e}^{-s/2} B(\mathrm{e}^s, t) \quad \text{for all } s, t \geq 0. \tag{2.2}$$

Indeed, one can check directly that U is a $C(\mathbf{R}_+)$-valued, stationary, symmetric diffusion. For every $t \geq 0$, $\{U_s(t)\}_{s\geq 0}$ solves the stochastic differential equation (1.1) of the Ornstein–Uhlenbeck type. Furthermore, the invariant measure of U is the Wiener measure.

The following well-known result is a useful localization tool.

Lemma 2.1. *For all bounded Borel sets $G \subseteq \mathbf{R}_+$ and $\Lambda \in \mathscr{B}$, $\mathrm{cap}_G(\Lambda) > 0$ if and only if with positive probability there exists $s \in G$ such that $U_s \in \Lambda$.*

Remark 2.2. The previous lemma continues to hold even when G is unbounded.

Proof. Without loss of much generality, we may – and will – assume that $G \subseteq [0, q]$ for some $q > 0$. Let $p_G(\Lambda)$ denote the probability that there exists $s \in G$ such that $U_s \in \Lambda$. Evidently, $\mathrm{cap}_G(\Lambda) \leq p_G(\Lambda)$. Furthermore,

$$\mathrm{cap}_G(\Lambda) = \int_0^q \mathrm{P}\left\{\exists s \in G \cap [0, \tau] : U_s \in \Lambda\right\} \mathrm{e}^{-\tau} \mathrm{d}\tau + \mathrm{e}^{-q} p_G(\Lambda). \tag{2.3}$$

Whence follow the bounds,

$$\mathrm{e}^{-q} p_G(\Lambda) \leq \mathrm{cap}_G(\Lambda) \leq p_G(\Lambda). \tag{2.4}$$

The lemma follows. □

Define $\{f^* \leq r\} := \{f \in C(\mathbf{R}_+) : f^* \leq r\}$, where

$$f^\star := \sup_{u \in [0,1]} |f(u)|. \tag{2.5}$$

The following is the main step in the proof of Theorem 1.1. It was announced earlier in the Abstract.

Theorem 2.3. *There exists $a > 1$ such that for all $r \in (0, 1)$ and all Borel sets $G \subseteq [0, 1]$,*

$$\frac{1}{a}\mathrm{K}_G(r^6) \exp\left(-\frac{\pi^2}{8r^2}\right) \leq \mathrm{cap}_G\left\{f^\star \leq r\right\} \leq a\mathrm{K}_G(r^6) \exp\left(-\frac{\pi^2}{8r^2}\right). \tag{2.6}$$

Remark 2.4. The constant a depends on G only through the fact that G is a subset of $[0, 1]$. Therefore, there exists $a > 1$ such that simultaneously for all Borel sets $F, G \subseteq [0, 1]$,

$$\frac{1}{a}\frac{\mathrm{K}_F(r^6)}{\mathrm{K}_G(r^6)} \leq \frac{\mathrm{cap}_F\{f^\star \leq r\}}{\mathrm{cap}_G\{f^\star \leq r\}} \leq a\frac{\mathrm{K}_F(r^6)}{\mathrm{K}_G(r^6)} \quad \text{for all } r \in (0, 1). \tag{2.7}$$

Remark 2.5. It turns out that for any fixed $\varepsilon > 0$, $\mathrm{cap}_{\mathbf{R}_+}$ and $\mathrm{cap}_{[0,\varepsilon]}$ are equivalent. To prove this, we can assume without loss of generality that $\varepsilon \in (0, 1)$. [This is because $\varepsilon \mapsto \mathrm{cap}_{[0,\varepsilon]}(\Lambda)$ is increasing.] Now, on the one hand, $\mathrm{cap}_{[0,\varepsilon]}(\Lambda) \leq \mathrm{cap}_{\mathbf{R}_+}(\Lambda)$. On the other hand,

$$\begin{aligned}\mathrm{cap}_{\mathbf{R}_+}(\Lambda) &\leq \int_0^\infty \sum_{0 \leq j \leq \sigma/\varepsilon} \mathrm{P}\left\{\exists s \in [j\varepsilon, (j+1)\varepsilon] : U_s \in \Lambda\right\} \mathrm{e}^{-\sigma}\mathrm{d}\sigma \\ &\leq \mathrm{P}\left\{\exists s \in [0, \varepsilon] : U_s \in \Lambda\right\} \cdot \int_0^\infty \frac{\sigma + 1}{\varepsilon}\mathrm{e}^{-\sigma}\mathrm{d}\sigma,\end{aligned} \tag{2.8}$$

by stationarity. In the notation of Lemma 2.1, the last term is equal to $(2/\varepsilon)p_{[0,\varepsilon]}(\Lambda)$, which is in turn dominated by $(2\mathrm{e}/\varepsilon)\mathrm{cap}_{[0,\varepsilon]}(\Lambda)$; confer with (2.4). Thus,

$$\frac{\varepsilon}{2\mathrm{e}}\mathrm{cap}_{\mathbf{R}_+}(\Lambda) \leq \mathrm{cap}_{[0,\varepsilon]}(\Lambda) \leq \mathrm{cap}_{\mathbf{R}_+}(\Lambda) \quad \text{for all } \Lambda \in \mathscr{B}. \tag{2.9}$$

This proves amply the claimed equivalence of $\mathrm{cap}_{[0,\varepsilon]}$ and $\mathrm{cap}_{\mathbf{R}_+}$.

According to the eigenfunction expansion of Chung [1],

$$\mu\left\{f^\star \le r\right\} \sim \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8r^2}\right) \quad (r \to 0). \tag{2.10}$$

Therefore, thanks to (2.4), Theorem 2.3 is equivalent to our next result.

Theorem 2.6. *Recall from (2.5) that $U_s^\star = \sup_{t\in[0,1]}|U_s(t)|$. Then, there exists a constant $a > 1$ such that for all $r \in (0,1)$ and all Borel sets $G \subseteq [0,1]$,*

$$\frac{1}{a}\mathrm{K}_G(r^6)\mu\left\{f^\star \le r\right\} \le \mathrm{P}\left\{\inf_{s\in G} U_s^\star \le r\right\} \le a\mathrm{K}_G(r^6)\mu\left\{f^\star \le r\right\}. \tag{2.11}$$

We will derive this particular reformulation of Theorem 2.3. The following result plays a key role in our analysis.

Proposition 2.7 (*[11, Proposition 2.1]*). *Let $\{X_t\}_{t\ge 0}$ denote planar Brownian motion. For every $r > 0$ and $\lambda \in (0,1]$ define*

$$\mathscr{D}_\lambda^r = \left\{(x,y) \in \mathbf{R}^2 : |x| \le r, \left|x(1-\lambda)^{1/2} + y\lambda^{1/2}\right| \le r\right\}. \tag{2.12}$$

Then there exists an $a \in (0, 1/2)$ such that for all $r > 0$ and $\lambda \in (0,1]$,

$$\mathrm{P}\left(\bigcap_{t\in[0,1]}\left\{X_t \in \mathscr{D}_\lambda^r\right\}\right) \le \frac{1}{a}\mu\left\{f^\star \le r\right\}\exp\left(-\frac{a\lambda^{1/3}}{r^2}\right). \tag{2.13}$$

Lemma 2.8. *There exists a constant $a \in (0,1)$ such that for all $1 \ge S > s > 0$,*

$$\mathrm{P}\left\{U_s^\star \le r, U_S^\star \le r\right\} \le \frac{1}{a}\mu\left\{f^\star \le r\right\}\exp\left(-\frac{a(S-s)^{1/3}}{r^2}\right), \tag{2.14}$$

valid for all $r \in (0,1)$.

Proof. Define $\lambda = 1 - \exp\{-(S-s)\}$. Then owing to (2.2) we can write

$$\begin{aligned} U_S(t) &= U_s(t)(1-\lambda)^{1/2} + \frac{B(\mathrm{e}^S, t) - B(\mathrm{e}^s, t)}{\left(\mathrm{e}^S - \mathrm{e}^s\right)^{1/2}} \\ \lambda^{1/2} &:= U_s(t)(1-\lambda)^{1/2} + V(t)\lambda^{1/2}. \end{aligned} \tag{2.15}$$

By the Markov properties of the Brownian sheet, $X_t := (U_s(t), V(t))$ defines a planar Brownian motion. Moreover, $\mathrm{P}\{U_s^\star \le r, U_S^\star \le r\} = \mathrm{P}(\cap_{t\in[0,1]}\{X_t \in \mathscr{D}_\lambda^r\})$. By Taylor's expansion, $1 - \exp(-x) \ge (x/2)$ $(x \in [0,1])$. Therefore, Proposition 2.7 completes the proof. □

Proof of Theorem 2.6 (*Lower Bound*). Let $k = \mathrm{K}_G(r^6)$, and choose maximal Kolmogorov points $s(1) < \cdots < s(k)$ such that $s(i+1) - s(i) \ge r^6$. Evidently, whenever $j > i$ we have $s(j) - s(i) \ge (j-i)r^6$. Now define

$$N_r = \sum_{i=1}^{k} \mathbf{1}_{\{U_{s(i)}^\star \le r\}}. \tag{2.16}$$

According to Lemma 2.8,

$$\begin{aligned}
\mathrm{E}\left[N_r^2\right] &= k\mu\left\{f^\star \le r\right\} + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \mathrm{P}\left\{U^\star_{s(i)} \le r, U^\star_{s(j)} \le r\right\} \\
&\le k\mu\left\{f^\star \le r\right\} + \frac{2}{a}\mu\left\{f^\star \le r\right\}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \exp\left(-\frac{a(s(j)-s(i))^{1/3}}{r^2}\right) \\
&\le k\mu\left\{f^\star \le r\right\} + \frac{2}{a}\mu\left\{f^\star \le r\right\}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \exp\left(-a(j-i)^{1/3}\right) \\
&\le Ak\mu\left\{f^\star \le r\right\}. \qquad (2.17)
\end{aligned}$$

Note that A is a positive and finite constant that does not depend on r. Also note that $\mathrm{E}[N_r] = k\mu\{f^\star \le r\}$. This and the Paley–Zygmund inequality ([8], Lemma 1.4.1, p. 72) together reveal that

$$\mathrm{P}\left\{\inf_{s\in G} U^\star_s \le r\right\} \ge \mathrm{P}\{N_r > 0\} \ge \frac{(\mathrm{E}[N_r])^2}{\mathrm{E}\left[N_r^2\right]} \ge \frac{k}{A}\mu\left\{f^\star \le r\right\}. \qquad (2.18)$$

The definition of k implies the lower bound in Theorem 2.6. □

Before proving the upper bound of Theorem 2.6 in complete generality, we first derive the following weak form:

Proposition 2.9. *There exists a finite constant $a > 1$ such that for all $r \in (0, 1)$,*

$$\mathrm{P}\left\{\inf_{s\in[0,r^6]} U^\star_s \le r\right\} \le a\mu\left\{f^\star \le r\right\}. \qquad (2.19)$$

Proof. Recall (2.16), and define

$$L(s; r) = \int_0^s \mathbf{1}_{\{U^\star_\nu \le r\}}\mathrm{d}\nu \quad \text{for all } s, r > 0. \qquad (2.20)$$

Let $\mathscr{F} := \{\mathscr{F}_s\}_{s\ge 0}$ denote the augmented filtration generated by the infinite-dimensional process $\{U_s\}_{s\ge 0}$. The latter process is Markov with respect to $\mathscr{F}$. Moreover,

$$\mathrm{E}\left[L(2r^6; r + r^3)|\mathscr{F}_s\right] \ge \int_s^{2r^6} \mathrm{P}\left\{U^\star_\nu \le r + r^3|\mathscr{F}_s\right\}\mathrm{d}\nu \cdot \mathbf{1}_{\{U^\star_s \le r\}}. \qquad (2.21)$$

As in (2.15), if $\nu > s$ are fixed, then we can write

$$\begin{aligned}
U_\nu(t) &= U_s(t)\mathrm{e}^{-(\nu-s)/2} + \frac{B(\mathrm{e}^\nu, t) - B(\mathrm{e}^s, t)}{(\mathrm{e}^\nu - \mathrm{e}^s)^{1/2}}\left[1 - \mathrm{e}^{-(\nu-s)}\right]^{1/2} \\
&:= U_s(t)\mathrm{e}^{-(\nu-s)/2} + V(t)\left[1 - \mathrm{e}^{-(\nu-s)}\right]^{1/2}. \qquad (2.22)
\end{aligned}$$

We emphasize, once again, that (U_s, V) is a planar Brownian motion. In addition, V is independent of $\mathscr{F}_s$, and $U^\star_\nu \le U^\star_s + V^\star[1 - \exp\{-(\nu - s)\}[^{1/2}$. Consequently, as long as $0 \le s \le r^6$ and $s < \nu < 2r^6$,

$$U^\star_\nu \le U^\star_s + 2^{1/2}r^3V^\star. \qquad (2.23)$$

[We have used the inequality $1 - \exp(-z) \le z$.] Therefore, for all $0 \le s \le r^6$,

$$\begin{aligned} M(s) &= \mathrm{E}\left[L(2r^6; r + r^3)|\mathscr{F}_s\right] \\ &\ge \int_s^{2r^6} \mathrm{P}\left\{V^\star \le 2^{-1/2}\right\} \mathrm{d}v \cdot \mathbf{1}_{\{U_s^\star \le r\}} \\ &= \mu\left\{f^\star \le 2^{-1/2}\right\}\left(2r^6 - s\right) \cdot \mathbf{1}_{\{U_s^\star \le r\}} \\ &\ge \mu\left\{f^\star \le 2^{-1/2}\right\} r^6 \cdot \mathbf{1}_{\{U_s^\star \le r\}}. \end{aligned} \tag{2.24}$$

Because $\{M(s)\}_{s\ge 0}$ is a martingale, we can apply Doob's maximal inequality to obtain the following:

$$\begin{aligned} \mathrm{P}\left\{\inf_{s\in[0,r^6]} U_s^\star \le r\right\} &\le \mathrm{P}\left\{\sup_{s\in[0,r^6]} M(s) \ge \mu\left\{f^\star \le 2^{-1/2}\right\} r^6\right\} \\ &\le \frac{\mathrm{E}\left[L(2r^6; r + r^3)\right]}{\mu\left\{f^\star \le 2^{-1/2}\right\} r^6} \\ &= \frac{2\mu\left\{f^\star \le r + r^3\right\}}{\mu\left\{f^\star \le 2^{-1/2}\right\}}. \end{aligned} \tag{2.25}$$

Thanks to (2.10), as $r \to 0$,

$$\frac{\mu\left\{f^\star \le r + r^3\right\}}{\mu\left\{f^\star \le r\right\}} \sim \exp\left(-\frac{\pi^2}{8}\left[\frac{1}{(r + r^3)^2} - \frac{1}{r^2}\right]\right) \to \exp(\pi^2/4). \tag{2.26}$$

Thus, the left-hand side is bounded ($r \in (0, 1)$), and the proposition follows. □

Proof of Theorem 2.6 (*Upper Bound*). Define $n = n(r)$ to be $\lfloor r^{-6}\rfloor$, and define $I(j; n)$ to be the interval $[j/n, (j + 1)/n)(j = 0, \ldots, n)$. Then, by stationarity and Proposition 2.9,

$$\mathrm{P}\left\{\inf_{s\in G} U_s^\star \le r\right\} \le \sum_{\substack{0\le j\le n:\\ I(j;n)\cap G\ne\varnothing}} \mathrm{P}\left\{\inf_{s\in I(j;n)} U_s^\star \le r\right\} \le a\mu\left\{f^\star \le r\right\} \mathrm{M}_n(G), \tag{2.27}$$

where $\mathrm{M}_n(G) = \#\{0 \le j \le n : I(j; n) \cap G \ne \varnothing\}$ defines the *Minkowski content* of G. In the companion to this paper [10, Proposition 2.7] we proved that $\mathrm{M}_n(G) \le 3\mathrm{K}_G(1/n)$. By monotonicity, the latter is at most $3\mathrm{K}_G(r^6)$, whence the theorem. □

3. Proof of Theorem 1.1 and Corollaries 1.2 and 1.4

We begin with some preliminary discussions. Define

$$\psi_H(G) := \int_1^\infty \frac{\mathrm{K}_G(H^6(s))}{sH^2(s)} \exp\left(-\frac{\pi^2}{8H^2(s)}\right) \mathrm{d}s, \qquad \sigma(r) := \mu\left\{f^\star \le r\right\}. \tag{3.1}$$

We follow Erdős [4] and define

$$\mathbf{e}_n := \exp\left(\frac{n}{\ln_+ n}\right), \qquad H_n := H(\mathbf{e}_n) \quad \text{for all } n \ge 1. \tag{3.2}$$

The "critical" function in (1.11) is $H(t) = \pi/(8 \ln_+ \ln_+ t)^{1/2}$. Thus, the fact that $\pi/8^{1/2} \in (1, 2)$, and a familiar argument [4, Equations 1.2 and 3.4], together allow us to assume without loss of generality that

$$\frac{1}{(\ln_+ n)^{1/2}} \leq H_n \leq \frac{2}{(\ln_+ n)^{1/2}} \quad \text{for all } n \geq 1. \tag{3.3}$$

From this we can conclude the existence of a constant $a > 1$ such that

$$\frac{1}{a} H_n^2 \mathbf{e}_{n+1} \leq \mathbf{e}_{n+1} - \mathbf{e}_n \leq a H_{n+1}^2 \mathbf{e}_n \quad \text{for all } n \geq 1. \tag{3.4}$$

According to our companion work [10, Eq. 2.8], for all $\varepsilon > 0$ sufficiently small,

$$\mathrm{K}_G(\varepsilon) \leq 6\mathrm{K}_G(2\varepsilon). \tag{3.5}$$

Because $\mathbf{e}_{n+1} \sim \mathbf{e}_n$ as $n \to \infty$, (2.10), (3.4) and (3.5) together imply that

$$\sum_{n=1}^{\infty} \mathrm{K}_G\left(H_n^6\right) \sigma\left(H_n\right) < \infty \quad \text{if and only if} \quad \psi_H(G) < \infty. \tag{3.6}$$

The following is the key step toward proving Theorem 1.1.

Proposition 3.1. *Let $H : \mathbf{R}_+ \to \mathbf{R}_+$ be decreasing and measurable. Then for all non-random Borel sets $G \subseteq [0, 1]$,*

$$\liminf_{t \to \infty} \left(\inf_{s \in G} \sup_{u \in [0,t]} |U_s(u)| - t^{1/2} H(t) \right) = \begin{cases} +\infty & \text{if } \psi_H(G) < \infty, \\ -\infty & \text{if } \psi_H(G) = \infty. \end{cases} \tag{3.7}$$

First we assume this proposition and derive Theorem 1.1. Then, we will tidy things up by proving the technical Proposition 3.1.

Let us recall (3.1).

Definition 3.2. We say that $\Psi_H(G) < \infty$ if we can decompose G as $G = \cup_{n=1}^{\infty} G_n$ – where $G_1, G_2, \ldots$ are closed – such that for all $n \geq 1$, $\psi_H(G_n) < \infty$. Else, we say that $\Psi_H(G) = \infty$.

Let us first rephrase Theorem 1.1 in the following convenient, and equivalent, form.

Proposition 3.3. *Let $H : \mathbf{R}_+ \to \mathbf{R}_+$ be decreasing and measurable and $G \subseteq [0, 1]$ be non-random and Borel. If $\Psi_H(G) < \infty$, then*

$$\inf_{s \in G} \liminf_{t \to \infty} \left(\sup_{u \in [0,t]} |U_s(u)| - t^{1/2} H(t) \right) = \infty \quad \mathrm{P}\text{-}a.s. \tag{3.8}$$

Else, the left-hand side is P-*a.s. equal to* $-\infty$.

Proof of Theorem 1.1 in the form of Proposition 3.3. First suppose that $\Psi_H(G)$ is finite. We can write $G = \cup_{n=1}^{\infty} G_n$, where the G_n's are closed and $\psi_H(G_n) < \infty$ for all $n \geq 1$. Then, according to Proposition 3.1,

$$\inf_{s \in G_n} \liminf_{t \to \infty} \left[\sup_{u \in [0,t]} |U_s(u)| - t^{1/2} H(t) \right]$$

$$\geq \liminf_{t\to\infty} \inf_{s\in G_n} \left[\sup_{u\in[0,t]} |U_s(u)| - t^{1/2}H(t) \right] = \infty. \tag{3.9}$$

This proves that $\inf_{s\in G} \liminf_{t\to\infty}(\sup_{u\in[0,t]} |U_s(u)| - t^{1/2}H(t)) = \infty$ a.s. [P].

For the converse portion suppose that $\Psi_H(G) = \infty$, and choose arbitrary non-random closed sets $\{G_n\}_{n=1}^{\infty}$ such that $\cup_{n=1}^{\infty} G_n = G$. By definition, $\psi_H(G_n) = \infty$ for some $n \geq 1$. Define for all $T \geq 1$,

$$\mathscr{S}_T := \left\{ s \in [0,1] : \inf_{t\geq T} \frac{\sup_{u\in[0,t]} |U_s(u)|}{t^{1/2}H(t)} \leq 1 \right\}. \tag{3.10}$$

Evidently, $\mathscr{S}_T$ is a random set for each $T \geq 0$. Moreover, the continuity of the Brownian sheet implies that with probability one, $\mathscr{S}_T$ is closed for all T; hence, so is $\mathscr{S}_T \cap G_n$. Because $\psi_H(G_n) = \infty$, Proposition 3.1 implies that almost surely, $\mathscr{S}_T \cap G_n \neq \varnothing$. Since $\{\mathscr{S}_T \cap G_n\}_{T=1}^{\infty}$ is a decreasing sequence of non-void compact sets, they have non-void intersection. That is, $(\cap_{T=1}^{\infty} \mathscr{S}_T) \cap G_n \neq \varnothing$ a.s. [P]. Replace H by $H - H^3$ to complete the proof of Proposition 3.3. □

Now we derive Proposition 3.1. This completes our proof of Theorem 1.1. Our proof is divided naturally into two halves.

Proof of Proposition 3.1 (*First Half*). Throughout this portion of the proof, we assume that $\psi_H(G) < \infty$.

Because $\mathbf{e}_{n+1} \sim \mathbf{e}_n$ as $n \to \infty$, Theorem 2.6 and Brownian scaling together imply that

$$\begin{aligned} \mathrm{P}\left\{ \inf_{s\in G} \sup_{u\in[0,\mathbf{e}_{n-1}]} |U_s(u)| \leq \mathbf{e}_n^{1/2} H_n \right\} &= \mathrm{P}\left\{ \inf_{s\in G} U_s^{\star} \leq H_n(\mathbf{e}_n/\mathbf{e}_{n-1})^{1/2} \right\} \\ &\leq a\mathrm{K}_G\left(H_n^6 \left[\frac{\mathbf{e}_n}{\mathbf{e}_{n-1}}\right]^3 \right) \sigma\left(H_n \left[\frac{\mathbf{e}_n}{\mathbf{e}_{n-1}}\right]^{1/2} \right). \end{aligned} \tag{3.11}$$

According to (3.5), $\mathrm{K}_G(\cdots) \leq 6\mathrm{K}_G(H_n^6)$ for all n large. This and (3.4) together imply that for all n large,

$$\begin{aligned} \mathrm{P}\left\{ \inf_{s\in G} \sup_{u\in[0,\mathbf{e}_{n-1}]} |U_s(u)| \leq \mathbf{e}_n^{1/2} H_n \right\} &\leq a\mathrm{K}_G\left(H_n^6\right) \sigma\left(H_n \left[1 + AH_{n+1}^2\right]^{1/2} \right) \\ &\leq a\mathrm{K}_G\left(H_n^6\right) \sigma\left(H_n \left[1 + AH_n^2\right] \right). \end{aligned} \tag{3.12}$$

In accordance with (2.10), for any fixed $c \in \mathbf{R}$,

$$\sigma\left(r + cr^3\right) = O(\sigma(r)) \quad (r \to 0). \tag{3.13}$$

Thus, for all $n \geq 1$,

$$\mathrm{P}\left\{ \inf_{s\in G} \sup_{u\in[0,\mathbf{e}_{n-1}]} |U_s(u)| \leq \mathbf{e}_n^{1/2} H_n \right\} \leq a\mathrm{K}_G\left(H_n^6\right) \sigma\left(H_n\right). \tag{3.14}$$

Because we are assuming that $\psi_H(G)$ is finite, (3.6) and the Borel–Cantelli lemma together imply that almost surely, $\inf_{s\in G}\sup_{u\in[0,\mathbf{e}_{n-1}]}|U_s(u)| > \mathbf{e}_n^{1/2}H_n$ for all but a finite number of n's. It follows from this and a standard monotonicity argument that

$$\psi_H(G) < \infty \Longrightarrow \liminf_{t\to\infty}\left[\inf_{s\in G}\sup_{u\in[0,t]}|U_s(u)| - t^{1/2}H(t)\right] > 0 \quad \text{a.s. [P].} \tag{3.15}$$

But if $\psi_H(G)$ were finite then $\psi_{H+H^3}(G)$ is also finite; compare (3.5) and (3.13). Thanks to (3.3), $\lim_{t\to\infty} t^{1/2}H^3(t) = \infty$. Therefore, the lim inf of the preceding display is infinity. This concludes the first half of our proof of Proposition 3.1. □

In order to prove the second half of Proposition 3.1 we assume that $\psi_H(G) = \infty$, recall (3.1), and define

$$\begin{aligned} L_n &:= \left\{\inf_{s\in G}\sup_{u\in[0,\mathbf{e}_n]}|U_s(u)| \le \mathbf{e}_n^{1/2}H_n\right\}, \\ f(z) &:= \mathrm{K}_G\left(z^6\right)\sigma(z). \end{aligned} \tag{3.16}$$

Lemma 3.4. *Define for all $j \ge i$, $\delta_{i,j} := H_j\lambda_{i,j}^{1/2} + H_i(\lambda_{i,j}-1)^{1/2}$, where $\lambda_{i,j} := \mathbf{e}_j/(\mathbf{e}_j - \mathbf{e}_i)$. Then, there exists $a > 1$ such that for all $j \ge i$,*

$$\mathrm{P}(L_j \mid L_i) \le a\mathrm{K}_G\left(\delta_{i,j}^6\right)\sigma\left(\delta_{i,j}\right). \tag{3.17}$$

Proof. Evidently, $\mathrm{P}(L_j \mid L_i)$ is at most

$$\begin{aligned} &\mathrm{P}\left\{\inf_{s\in G}\sup_{u\in[\mathbf{e}_i,\mathbf{e}_j]}|U_s(u)| \le \mathbf{e}_j^{1/2}H_j \,\middle|\, L_i\right\} \\ &= \mathrm{P}\left\{\inf_{s\in G}\sup_{u\in[\mathbf{e}_i,\mathbf{e}_j]}|U_s(u) - U_s(\mathbf{e}_i) + U_s(\mathbf{e}_i)| \le \mathbf{e}_j^{1/2}H_j \,\middle|\, L_i\right\} \\ &\le \mathrm{P}\left\{\inf_{s\in G}\sup_{u\in[\mathbf{e}_i,\mathbf{e}_j]}|U_s(u) - U_s(\mathbf{e}_i)| \le \mathbf{e}_j^{1/2}H_j + \mathbf{e}_i^{1/2}H_i\right\}. \end{aligned} \tag{3.18}$$

We have appealed to the Markov properties of the Brownian sheet in the last line. Because $u \mapsto U_\bullet(u)$ is a $C(\mathbf{R}_+)$-valued Brownian motion,

$$\begin{aligned} \mathrm{P}(L_j \mid L_i) &\le \mathrm{P}\left\{\inf_{s\in G}\sup_{u\in[0,\mathbf{e}_j-\mathbf{e}_i]}|U_s(u)| \le \mathbf{e}_j^{1/2}H_j + \mathbf{e}_i^{1/2}H_i\right\} \\ &= \mathrm{P}\left\{\inf_{s\in G} U_s^\star \le \delta_{i,j}\right\}. \end{aligned} \tag{3.19}$$

Theorem 2.6 completes the proof. □

Our forthcoming estimates of $\mathrm{P}(L_j \mid L_i)$ rely on the following elementary bound; see, for example, our earlier work [9, eq. 8.30]: Uniformly for all integers $j > i$,

$$\mathbf{e}_j - \mathbf{e}_i \ge \mathbf{e}_i\left(\frac{j-i}{\ln i}\right)(1+o(1)) \quad (i \to \infty). \tag{3.20}$$

Lemma 3.5. *There exist $i_0 \geq 1$ and a finite $a > 1$ such that for all $i \geq i_0$ and $j \geq i + \ln^{19}(j)$,*

$$\mathrm{P}(L_j \mid L_i) \leq a\mathrm{P}(L_j). \tag{3.21}$$

Proof. Thanks to (3.20), the following holds uniformly over all $j > i + \ln^{19}(j)$:

$$\frac{\mathbf{e}_j}{\mathbf{e}_i} \geq 1 + \frac{(j-i)(1+o(1))}{\ln i} \geq 1 + (1+o(1))\frac{\ln^{19}(j)}{\ln i} \geq (1+o(1))\ln^{18}(j), \tag{3.22}$$

as $i \to \infty$. It follows from (3.3) that $\mathbf{e}_j/\mathbf{e}_i \geq (1+o(1))H_j^{-36}$, uniformly in $j > i + \ln^{19}(j)$ as $i \to \infty$. Thus, uniformly over all $j > i + \ln^{19}(j)$,

$$\begin{aligned} \lambda_{i,j}^{1/2} &= \frac{1}{\left[1-(\mathbf{e}_i/\mathbf{e}_j)\right]^{1/2}} \leq \frac{1}{\left[1-(1+o(1))H_j^{36}\right]^{1/2}} = 1 + O\left(H_j^3\right), \\ H_i(\lambda_{i,j}-1)^{1/2} &= O\left(H_j^3\right) \quad (i \to \infty). \end{aligned} \tag{3.23}$$

Lemma 3.4 guarantees then that uniformly over all $j > i + \ln^{19}(j)$, $\delta_{i,j} \leq H_j + O(H_j^3)$, and the big-$O$ and little-o terms do not depend on the j's in question. The lemma follows from this, Eqs. (3.5) and (3.13), and Theorem 2.6. □

Lemma 3.6. *There exist $i_1 \geq 1$ and $a \in (0,1)$ such that for all $i \geq i_1$ and $j \in [i + \ln(i), i + \ln^{19}(j))$,*

$$\mathrm{P}(L_j \mid L_i) \leq \frac{1}{aj^a}. \tag{3.24}$$

Proof. Eqs. (3.20) and (3.3) together imply that uniformly for all $j \geq i + \ln(i)$, $(\mathbf{e}_i/\mathbf{e}_j) \leq \frac{1}{2} + o(1)(i \to \infty)$. This is equivalent to the existence of a constant $A_{(3.25)}$ such that for all (i,j) in the range of the lemma,

$$\max\left(\lambda_{i,j}^{1/2}, (\lambda_{i,j}-1)^{1/2}\right) \leq a. \tag{3.25}$$

Thanks to (3.3), we can enlarge the last constant a, if necessary, to ensure that for all (i,j) in the range of this lemma, $H_i \leq aH_j$. Therefore, Lemma 3.4 then implies that $\delta_{i,j} = O(H_j)$, and the big-O term does not depend on the range of j's in question. Because $G \subseteq [0,1]$,

$$\mathrm{K}_G(\varepsilon) \leq \mathrm{K}_{[0,1]}(\varepsilon) \sim \frac{1}{\varepsilon} \quad (\varepsilon \to 0). \tag{3.26}$$

Thus, Lemma 3.4 ensures that $\mathrm{P}(L_j \mid L_i) \leq a\delta_{i,j}^{-6}\sigma(\delta_{i,j})$. Near the origin, the function $\delta \mapsto \delta^{-6}\sigma(\delta)$ is increasing. Because we have proved that over the range of (i,j) of this lemma $\delta_{i,j} = O(H_j)$, Eq. (2.10) asserts the existence of a universal $\alpha > 1$ such that $\mathrm{P}(L_j \mid L_i)$ is at most $\alpha H_j^{-6}\exp(-\alpha^{-1}H_j^{-2})$. Eq. (3.3) then completes our proof. □

Lemma 3.7. *There exist $i_2 \geq 1$ and $a > 1$ such that for all $i \geq i_2$ and $j \in (i, i + \ln i)$,*

$$\mathrm{P}(L_j \mid L_i) \leq a\exp\left(-\frac{j-i}{a}\right). \tag{3.27}$$

Proof. By (3.20), $(\mathbf{e}_i/\mathbf{e}_j) \leq 1 - (1+o(1))(j-i)\ln^{-1}(i)(i \to \infty)$, where the little-$o$ term does not depend on $j \in (i, i+\ln i)$. Similarly, $(\mathbf{e}_j/\mathbf{e}_i) \geq 1 + (1+o(1))(j-i)\ln^{-1}(i)$. Thus, as $i \to \infty$,

$$\begin{aligned} \lambda_{i,j}^{1/2} &= \frac{1}{\left[1-(\mathbf{e}_i/\mathbf{e}_j)\right]^{1/2}} \leq (1+o(1))\left(\frac{\ln i}{j-i}\right)^{1/2} \leq \frac{2+o(1)}{(j-i)^{1/2}H_j}, \\ (\lambda_{i,j}-1)^{1/2} &= \frac{1}{\left[(\mathbf{e}_j/\mathbf{e}_i)-1\right]^{1/2}} \leq (1+o(1))\left(\frac{\ln i}{j-i}\right)^{1/2} \leq \frac{2+o(1)}{(j-i)^{1/2}H_j}, \end{aligned} \tag{3.28}$$

by (3.3). Once again, the little-o terms are all independent of $j \in (i, i+\ln i)$. Because $H_i = O(H_j)$ uniformly for all (i,j) in the range considered here, Lemma 3.4 implies that $\delta_{i,j} = O((j-i)^{-1/2})$, uniformly for all $j \in (i, i+\ln i)$. Eq. (3.26) bounds the first term on the right-hand side; (2.10) bounds the second. This and (3.3) together prove the existence of a constant $\alpha > 1$ such that for all $i \geq i_2$ and all $j \in (i, i+\ln i)$, $\mathrm{P}(L_j \mid L_i) \leq \alpha(j-i)^3 \exp\{-(j-i)/\alpha\}$. The lemma follows. □

Proof of Proposition 3.1 (*Second Half*). According to Theorem 2.6, for all n large enough, $\mathrm{P}(L_n) \geq af(H_n)$. Because $\psi_H(G) = \infty$, the latter estimate and (3.6) together imply that

$$\sum_{i=1}^{\infty} \mathrm{P}(L_i) = \infty. \tag{3.29}$$

Thus, our derivation is complete once we demonstrate the following:

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n-1}\sum_{j=i}^{n} \mathrm{P}(L_i \cap L_j)}{\left(\sum_{i=1}^{n} \mathrm{P}(L_i)\right)^2} < \infty. \tag{3.30}$$

See [2]. In fact, the preceding display holds with a lim sup in place of the lim inf. This fact follows from combining, using standard arguments, Lemmas 3.5 through 3.7.

Indeed, let $I := \max(3, i_1, i_2, i_3)$ and $s_n := \sum_{i=1}^{n} \mathrm{P}(L_i)$. Lemma 3.5 ensures that

$$\sum_{i=I}^{n-1}\sum_{\substack{j=i \\ j>i+\ln^{19}(j)}}^{n} \mathrm{P}(L_j \cap L_i) = O\left(s_n^2\right). \tag{3.31}$$

By Lemma 3.6,

$$\begin{aligned} \sum_{i=I}^{n-1}\sum_{\substack{j=i \\ j\in\left(i+\ln(i), i+\ln^{19}(j)\right]}}^{n} \mathrm{P}(L_j \cap L_i) &\leq \frac{1}{a} \sum_{i=I}^{n-1}\sum_{\substack{j=i \\ j\in\left(i+\ln(i), i+\ln^{19}(j)\right]}}^{n} j^{-a}\mathrm{P}(L_i) \\ &= \sum_{i=I}^{n} O\left(\frac{\ln^{19}(i)}{i^a}\right)\mathrm{P}(L_i) \\ &= O\left(s_n\right). \end{aligned} \tag{3.32}$$

The big-O terms do not depend on the variables (j, n).

Finally, Lemma 3.7 implies that

$$\sum_{\substack{i=I \\ j\in(i,i+\ln i]}}^{n-1}\sum_{j=i}^{n} \mathrm{P}(L_j \cap L_i) \le a \sum_{i=1}^{n}\sum_{j=i}^{\infty} \mathrm{P}(L_i)\mathrm{e}^{(j-i)/a} = O(s_n). \tag{3.33}$$

We have seen already that $s_n \to \infty$. Thus, (3.31)–(3.33) together imply (3.30), and hence the theorem. More precisely, we have proved so far that

$$\psi_H(G) = \infty \Longrightarrow \liminf_{t\to\infty}\left[\inf_{s\in G}\sup_{u\in[0,t]}|U_s(u)| - t^{1/2}H(t)\right] < 0 \quad \text{a.s. [P]}. \tag{3.34}$$

Replace H by $H + H^3$ to deduce that the preceding lim inf is in fact $-\infty$. This completes our proof of Proposition 3.1. □

We conclude this section by proving the remaining Corollaries 1.2 and 1.4.

Proof of Corollary 1.2. By definition, $\mathscr{L}(H)$ holds q.s. if and only if $\mathrm{cap}_{\mathbf{R}_+}((\mathscr{L}(H))^c) = 0$. Thanks to Theorem 1.1, this condition is equivalent to the existence of a non-random "closed-denumerable" decomposition $\mathbf{R}_+ = \cup_{n=1}^{\infty} G_n$ such that for all $n \ge 1$, $\psi_H(G_n) < \infty$. But one of the G_n's must contain a closed interval that has positive length. Therefore, by the translation-invariance of $G \mapsto \mathrm{K}_G(r)$, there exists $\varepsilon \in (0, 1)$ such that $\psi_H([0, \varepsilon]) < \infty$.

Conversely, if $\psi_H([0, \varepsilon])$ is finite, then we can define G_n to be $[(n-1)\varepsilon, n\varepsilon]$ $(n \ge 1)$ to find that $\psi_H(G_n) = \psi_H([0, \varepsilon]) < \infty$. Theorem 1.1 then proves that $\mathrm{cap}_{\mathbf{R}_+}((\mathscr{L}(H))^c) = 0$ if and only if there exists $\varepsilon > 0$ such that $\psi_H([0, \varepsilon]) < \infty$. Because $\mathrm{K}_{[0,\varepsilon]}(r) \sim \varepsilon/r$ as $r \to 0$, the corollary follows. □

Proof of Corollary 1.4. We can change the variables to deduce that $\psi_{H_\nu}(G)$ is finite if and only if $\int_1^\infty \mathrm{K}_G(1/s)s^{-1-(\nu/3)}\mathrm{d}s$ converges. This and Proposition 2.8 of our companion work [10] together imply that

$$\inf\{\nu > 0 : \psi_{H_\nu}(G) < \infty\} = 2 + 3\overline{\dim}_{\mathrm{M}} G, \tag{3.35}$$

where $\overline{\dim}_{\mathrm{M}}$ denotes the (upper) Minkowski dimension [13]. By regularization [13, p. 81],

$$\inf\{\nu > 0 : \Psi_{H_\nu}(G) < \infty\} = 2 + 3\dim_{\mathrm{P}} G. \tag{3.36}$$

Theorem 1.1 now implies Corollary 1.4. □

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